NEW-EXTENDED RAYLEIGH DISTRIBUTION - PROPERTIES AND ESTIMATION

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ABSTRACT

In this paper, we propose a New Extended Rayleigh Distribution (NERD) and explore its theoretical and practical significance in statistical modelling. The NERD is formulated as a flexible generalization of the classical Rayleigh distribution by introducing an additional shape parameter, thereby enhancing its ability to model a wider variety of real-world phenomena, particularly in lifetime and reliability data analysis. We rigorously define the distribution and derive several important statistical properties, including the r-th moments, moment generating function (MGF), characteristic function (CF), and cumulant generating function (CGF). These properties not only provide insight into the behaviour of the distribution but also facilitate its application in inferential and applied statistics. To estimate the unknown parameters of the NERD, we employ the Maximum Likelihood Estimation (MLE) method.

Key words: Distribution; Rayleigh; Reliability; Hazard Function; Maximum Likelihood Estimation.

I. INTRODUCTION

Statistical distribution and dissemination play a vital role in analyzing and assessing the authentic scenario of the real world. Indeed, the fact that moderate number of distributions has been developed. There is always scope to developing distributions, analyzing their properties that are more flexible or to adjust real world scenarios. The researchers are continually urging for establishing new and more flexible distributions. For that reason, many new distributions have been emerged and studies. In current works, new distributions are outlined by means of including one or more parameters to a distribution functions. Such an addition of parameters makes the ensuing distribution richer for modeling life time data. The generalized distributions have been invented to characterize different phenomena. These generalized distributions are also having a greater number of parameters. Johnson *et al.* (1994) clearly emphasized the four parameter distributions that are much essential for the workable situations. Many authors were not having crystal-clear opinion on whether three parameters or more than three were necessary for a better analysis. Adding too many parameters to the distribution may not help for a successful inference. "This idea to add a shape parameter gave rise to several models, including the proportional hazards model (PHM), the power transformed model (PTM), the proportional odds model (POM), and the proportional reversed hazard model (PRHM)". In recent years, many of the distributions have been developed based on the beta distribution. The cumulative distribution function (cdf) of generalized beta distribution for the random variable (X) is defined by,

$$F(x) = \frac{1}{B(\alpha,\beta)} \int_0^{G(x)} t^{\alpha-1}(1-t)^{\beta-1}\,dt;\ t>0,\alpha,\beta>0 \tag{1}$$

where G(x) is the Cumulative Distribution Function of any other distribution. The above function is an example of invention of new distribution by addition of parameter(s). The following authors who studied the above class of distributions are “Eugene *et al.* (2002), Nadarajah and Kotz (2004, 2006), Famoye *et al.* (2005), Kozubowski and Nadarajah (2008), Akinsete *et al.* (2008), Akinsete and Lowe (2009)”. By studying all these articles, we decided to develop a new generalized distribution.

Introducing a flexible distribution by adding a parameter that acts as both a shape parameter and a scale parameter ensures accuracy when fitting datasets in medicine, reliability engineering, and finance. This study has mainly focused on the shape or scale parameters which can accurately determine the new family distributions. The level of flexibility may increase due to introduce both parameters shape and scale.

For generalizing the existing probability distributions, a new family distribution is introduced by adding scale, shape and location parameters. One of the most interesting ways to add shape parameters to an existing distribution is exponentiation. Augmented family precursors from Mudholkar and Srivastava (1993) are defined by the following cdf.

$$G(x;c,\xi) = F(x;\xi)^c, c,\xi > 0, x \in R, \quad (2)$$

where c is considered as the additional shape parameter.

The prominent authors like Marshall and Olkin initiated a novel introducing a single-scale parameter to a family distribution. Marshall-Olkin's (MO) (1997) cdf of family is as follows

$$G(x,\sigma,\xi) = \frac{F(x;\xi)}{\sigma+(1-\sigma)F(x;\xi)}, \sigma,\xi > 0, x \in R, \quad (3)$$

where σ is taken as an extra parameter.

Cordeiro and Castro proposed (2011) developed a new family of generalized distributions defined by

$$G(x;k,m,\xi) = 1\text{-}\left(1-F(X;\xi)^k\right)^m \; ,k,m,\xi > 0, x \in R, \quad (4)$$

Undeniably the scale and shape parameters increase the degree of flexibility in the family distribution, but on the other hand the huge increase in the parameters, may also affect the calculation of mathematical functions making things more complex and complicated.

Attempts to introduce probability distributions with greater flexibility by introducing an additional parameter that is both a scale parameter and a shape parameter, thus providing higher accuracy for fitting to real data in applications such as reliability engineering, medicine, finance. Therefore, this paper aims to propose the latest method to introduce advanced statistical distributions. The proposed family can be named as the new extended Rayleigh Distribution (NERD) family. Let X be a random variable follows the proposed family then the cumulative distribution function is given by

$$G(x;\gamma;\delta) = 1-\left[\frac{1-F(x,\delta)^2}{1-(1-\gamma)F(x,\delta)^2}\right]^{\gamma} \; , \gamma>0, \; \delta > 0, x \in R \quad (5)$$

II. NEW EXTENDED RAYLEIGH DISTRIBUTION

The CDF of Rayleigh distribution is given by $F(x) = 1 - e^{-x^2/2\sigma^2}$

Then Pdf is given by

$$f(x) = \frac{d}{dx}\left[F(x)\right]$$

$$= \frac{d}{dx}\left[1 - e^{-x^2/2\sigma^2}\right]$$

$$= 0 - e^{-x^2/2\sigma^2}\frac{d}{dx}\left(\frac{-x^2}{2\sigma^2}\right)$$

$$f(x) = \frac{x}{\sigma^2}e^{-x^2/2\sigma^2}$$

The New extended Rayleigh distribution function is given by

$$G(x) = 1 - \left[\frac{1 - \left[F(x,\sigma)\right]^2}{1 - (1-\theta)\left[F(x,\sigma)\right]^2}\right]^\theta$$

Substitute F(x) value in above equation

$$G(x,\sigma,\theta) = 1 - \left[\frac{1 - \left(1 - e^{-x^2/2\sigma^2}\right)^2}{1 - (1-\theta)\left(1 - e^{-x^2/2\sigma^2}\right)^2}\right]^\theta$$

The probability density function g(x) is obtain by derivate the distribution function G(x) with respect to X i.e.

$$g(x) = \frac{d}{dx}\left[G(x)\right]$$

Substituting G(x) value

$$g(x) = \frac{d}{dx}\left[1 - \left[\frac{1 - \left(1 - e^{-x^2/2\sigma^2}\right)^2}{1 - (1-\theta)\left(1 - e^{-x^2/2\sigma^2}\right)^2}\right]^\theta\right]$$

$$= -\theta\left[\frac{1 - \left(1 - e^{-x^2/2\sigma^2}\right)^2}{1 - (1-\theta)\left(1 - e^{-x^2/2\sigma^2}\right)^2}\right]^{\theta-1}\frac{d}{dx}\left[\frac{1 - \left(1 - e^{-x^2/2\sigma^2}\right)^2}{1 - (1-\theta)\left(1 - e^{-x^2/2\sigma^2}\right)}\right]$$

$$g(x) = \frac{2\theta^2 x e^{-x^2/2\sigma^2}\left(1 - e^{-x^2/2\sigma^2}\right)\left[1 - \left(1 - e^{-x^2/2\sigma^2}\right)^2\right]^{\theta-1}}{\sigma^2\left[1 - (1-\theta)\left(1 - e^{-x^2/2\sigma^2}\right)^2\right]^{\theta+1}}$$

RELIABILITY FUNCTION:

$$R(x) = 1 - G(x)$$

$$=1-\left[1-\left[\frac{1-\left(1-e^{-x^2/2\sigma^2}\right)}{1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)}\right]^{\theta}\right]$$

$$=\left[\frac{1-\left(1-e^{-x^2/2\sigma^2}\right)^2}{1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2}\right]^{\theta}$$

$$R(x)=\left[\frac{1-\left(1-e^{-x^2/2\sigma^2}\right)^2}{1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2}\right]^{\theta}$$

HAZARD FUNCTION:

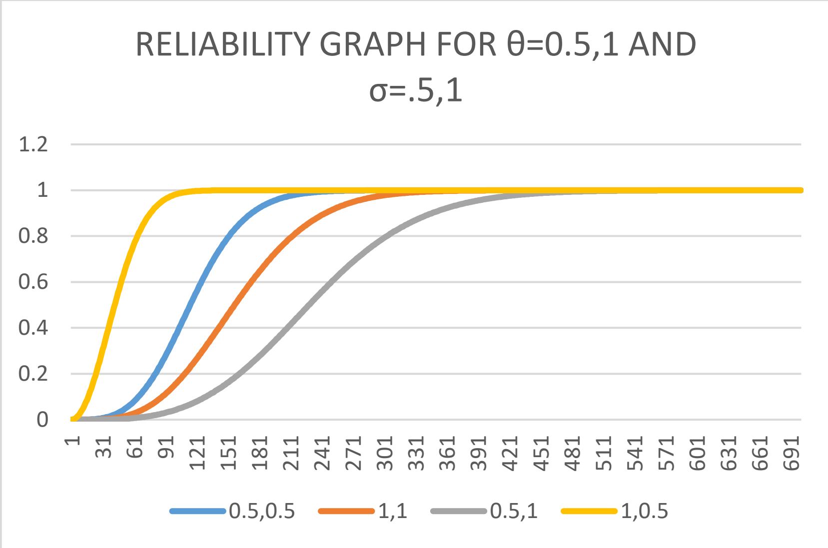
$$h(x)=\frac{g(x)}{R(x)}$$

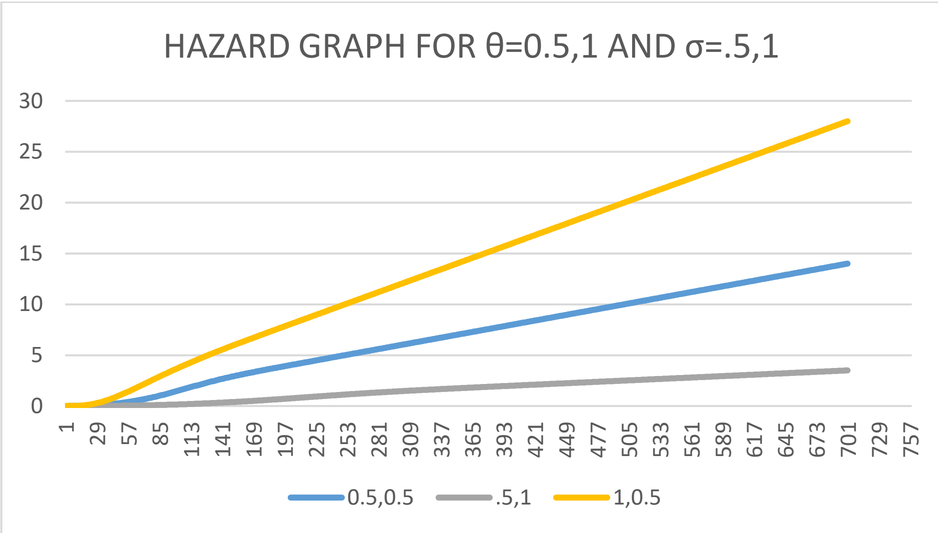
$$=\frac{2\theta^2 x e^{-x^2/2\sigma^2}\left(1-e^{-x^2/2\sigma^2}\right)\left[1-\left(1-e^{-x^2/2\sigma^2}\right)^2\right]^{\theta-1}}{\sigma^2\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]^{\theta+1}}\times\frac{\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]^{\theta}}{\left[1-\left(1-e^{-x^2/2\sigma^2}\right)^2\right]^{\theta}}$$

$$=\frac{2\theta^2 x e^{-x^2/2\sigma^2}\left(1-e^{-x^2/2\sigma^2}\right)}{\sigma^2\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]\left[1-\left(1-e^{-x^2/2\sigma^2}\right)^2\right]}$$

$$h(x)=\frac{2\theta^2 x\left(1-e^{-x^2/2\sigma^2}\right)}{\sigma^2\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]\left[2-e^{-x^2/2\sigma^2}\right]}$$

GRAPHICAL REPRESENTATION OF CDF, PDF, RELIABILITY AND HAZARD FUNCTION





With different values of the parameters the reliability and hazard function graphs are given above based on the hazard graph NE-R Distribution is an increasing failure model.

III. PROPERTIES OF NEW -EXTENDED RAYLEIGH DISTRIBUTION

MODE:

$$g(x)=\frac{2\theta^2 xe^{-x^2/2\sigma^2}\left(1-e^{-x^2/2\sigma^2}\right)\left[1-\left(1-e^{-x^2/2\sigma^2}\right)^2\right]^{\theta-1}}{\sigma^2\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]^{\theta+1}}$$

Multiplying the g(x) with log on both sides then we have

$$\log(g(x))=\log 2+2\log\theta+\log x-\frac{x^2}{2\sigma^2}+\log\left(1-e^{-x^2/2\sigma^2}\right)+(\theta-1)\log\left[1-\left(1-e^{-x^2/2\sigma^2}\right)^2\right]+2\log\sigma+(\theta+1)\log\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]$$

Differentiate the above log function with respect to x and equate it to 0

$$\frac{d}{dx}\left[\log g(x)\right]=0+0+\frac{1}{x}-\frac{x}{\sigma^2}+\frac{xe^{-x^2/2\sigma^2}}{\sigma^2\left(1-e^{-x^2/2\sigma^2}\right)}+\frac{(\theta-1)-2\left(1-e^{-x^2/2\sigma^2}\right)xe^{-x^2/2\sigma^2}}{\sigma^2\left[1-\left(1-e^{-x^2/2\sigma^2}\right)^2\right]}+\frac{(\theta+1)(1-\theta)\left(-2\left(1-e^{-x^2/2\sigma^2}\right)\right)xe^{-x^2/2\sigma^2}}{\sigma^2\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]}$$

$$=\frac{1}{x}-\frac{x}{\sigma^2}+\frac{xe^{-x^2/2\sigma^2}}{\sigma^2\left(1-e^{-x^2/2\sigma^2}\right)}-\frac{2\left(1-e^{-x^2/2\sigma^2}\right)xe^{-x^2/2\sigma^2}}{\sigma^2}\left[\frac{(\theta-1)}{\left(1-\left(1-e^{-x^2/2\sigma^2}\right)^2\right)}+\frac{(\theta+1)(1-\theta)}{\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]}\right]$$

$$\Rightarrow\frac{1}{x}-\frac{x}{\sigma^2}\left[1-2\left(1-e^{-x^2/2\sigma^2}\right)^2\right]\left[\frac{\left(\theta-\theta^2\right)+(2\theta)(\theta-1)\left(1-e^{-x/2\sigma^2}\right)^2}{\left[1-\left(1-e^{-x^2/2\sigma^2}\right)^2\right]\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]}\right]=0$$

MOMENTS:

The r^{th} moment of the New extended Rayleigh distribution is given by

$$\mu_r^{'}=\int_{-\infty}^{\infty}x^r g(x,\sigma,\theta,)dx$$

Substituting the value of $g(x,\sigma,\theta)$ in above equation then we get

$$\mu_r^{'}=\int_{-\infty}^{\infty}\frac{x^r 2\theta^2 xe^{-x^2/2\sigma^2}\left(1-e^{-x^2/2\sigma^2}\right)\left(1-\left(1-e^{-x^2/2\sigma^2}\right)\right)^{\theta-1}}{\sigma^2\left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]^{\theta+1}}dx$$

$$\mu_r^{'}=2\theta^2\int_{-\infty}^{\infty}x^r\frac{xe^{-x^2/2\sigma^2}}{\sigma^2}\frac{F(x,\sigma)\left[1-F(x,\sigma)^2\right]^{\theta-1}}{\left[1-(1-\theta)F(x,\sigma)^2\right]^{\theta+1}}dx\rightarrow 1$$

By using the expansion

$$\frac{1}{(1-x)^n}=\sum_{i=0}^{\infty}\binom{i+n-1}{n-1}x_i$$

Let $x=(1-\theta)\left(F(x,\sigma)\right)^2$ and $n=\theta+1$

Then

$$\frac{1}{\left[1-(1-\theta)\left(F(x,\sigma)\right)^2\right]^{\theta+1}}=\sum_{i=0}^{\infty}\binom{i+\theta+1-1}{\theta+1-1}\left[(1-\theta)\left(F(x,\sigma)\right)^2\right]^i$$

$$\frac{1}{\left[1-(1-\theta)\left(F(x,\sigma)\right)^2\right]^{\theta+1}}=\sum_{i=0}^{\infty}\binom{i+\theta}{\theta}(1-\theta)^i\left(F(x,\sigma)\right)^{2i}\rightarrow 2$$

Also using the series representation

$$(1-y)^m=\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^j$$

Let $y = \left[F(x,\sigma)\right]^2$ and $m = \theta - 1$

Then $$\left[1-\left(F(x,\sigma)\right)^2\right]^{\theta-1} = \sum_{j=0}^{\theta-1}\binom{\theta-1}{j}(-1)^j\left[F(x,\sigma)\right]^{2j}$$

$$\left[1-\left[F(x,\sigma)^2\right]\right]^{\theta-1} = \sum_{j=0}^{\theta-1}\binom{\theta-1}{j}(-1)^j\left[F(x,\sigma)\right]^{2j} \rightarrow 3$$

Using (1), (2) and (3) we have

$$\mu_r^{'} = 2\theta^2\int_{-\infty}^{\infty} x^r f(x,\sigma)F(x,\sigma)\sum_{j=0}^{\theta-1}\binom{\theta-1}{j}(-1)^j\left[F(x,\sigma)\right]^{2j}\sum_{i=0}^{\infty}\binom{i+\theta}{\theta}(1-\theta)^i\left[F(x,\sigma)\right]^{2i}dx$$

$$\mu_r^{'} = 2\theta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\theta-1}\binom{\theta-1}{j}\binom{i+\theta}{\theta}(-1)^j(1-\theta)^i S_{r,2(i+j)+1}$$

Where $$S_{r,2(i+j)+1} = \int_{-\infty}^{\infty} x^r f(x,\sigma)F(x,\sigma)^{2(i+j)+1}dx$$

Therefore $$\mu_r^{'} = 2\theta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\theta-1}\binom{\theta-1}{j}\binom{i+\theta}{\theta}(-1)^j(1-\theta)^i S_{r,2(i+j)+1}$$

MOMENT GENERATING FUNCTION:

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{'}$$

$$= 2\theta^2\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i=o}^{\infty}\sum_{j=0}^{\theta-1}\binom{i+\theta}{\theta}\binom{\theta-1}{j}(-1)^j(1-\theta)^i S_{r,2(i+j)+1}$$

$$M_x(t) = 2\theta^2\sum_{r=o}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\theta-1}\frac{t^r}{r!}\binom{i+\theta}{\theta}\binom{\theta-1}{j}(1-\theta)^i(-1)^j S_{r,2(i+j)+1}$$

CHARACTERISTIC FUNCTION:

$$\phi_x(t) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\mu_r^{'}$$

$$= 2\theta^2 \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{i=o}^{\infty} \sum_{j=0}^{\theta-1} \binom{i+\theta}{\theta} \binom{\theta-1}{j} (1-\theta)^i (-1)^j S_{r,2(i+j)+1}$$

$$\phi_x(t) = 2\theta^2 \sum_{r=o}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\theta-1} \frac{(it)^r}{r!} \binom{i+\theta}{\theta} \binom{\theta-1}{j} (1-\theta)^i (-1)^j S_{r,2(i+j)+1}$$

CUMMULATIVE GENERATING FUNCTION:

$$K_x(t) = L_n(M_x(t))$$

$$= \ln\left[2\theta^2 \sum_{r=o}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\theta-1} \frac{(t)^r}{r!} \binom{i+\theta}{\theta} \binom{\theta-1}{j} (1-\theta)^i (-1)^j S_{r,2(i+j)+1} \right]$$

$$K_x(t) = \ln\left[2\theta^2 \sum_{r=o}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\theta-1} \frac{(t)^r}{r!} \binom{i+\theta}{\theta} \binom{\theta-1}{j} (1-\theta)^i (-1)^j S_{r,2(i+j)+1} \right]$$

IV. MAXIMUM LIKELI HOOD ESTIMATIONS:

$$L = \prod_{i=1}^{n} g(x_i)$$

Substituting g(x) value then we get

$$L = \frac{2^n \theta^{2n} \left(\prod_{i=1}^{n} x_i\right) \left(\prod_{i=1}^{n} e^{-x^2/2\sigma^2}\right) \left(\prod_{i=1}^{n} \left(1-e^{-x^2/2\sigma^2}\right)\right) \left(\prod_{i=1}^{n} \left[1-\left(1-e^{-x^2/2\sigma 2}\right)^{\theta-1}\right]\right)}{\sigma^{2n} \prod_{i=1}^{n} \left[1-(1-\theta)\left(1-e^{-x^2/2\sigma^2}\right)^2\right]^{\theta+1}}$$

Now multiplying the above equation with log on both sides, we get

$$\log L = \log\left[\frac{2^n \theta^{2n} \prod_{i=1}^{n} x_i \prod_{i=1}^{n} e^{-x^2/2\sigma^2} \prod_{i=1}^{n} (a) \prod_{i=1}^{n} \left[1-(a)^2\right]^{\theta-1}}{\sigma^{2n} \prod_{i=1}^{n} \left[1-(1-\theta)(a)^2\right]^{\theta+1}} \right] \rightarrow 1 \quad \text{Where,} \quad a = \left(1-e^{-x^2/2\sigma^2}\right)$$

$$= n\log 2 + 2n\log\theta + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \left(\frac{-x_i^2}{2\sigma^2}\right) + \sum_{i=1}^{n} \log(a) + \sum_{i=1}^{n} (\theta-1)\log\left[1-(a)^2\right] - 2n\log\sigma - (\theta+1)\sum_{i=1}^{n} \log\left[1-(1-\theta)(a)^2\right] \rightarrow A$$

Now partial differentiate A with the parameters θ and σ

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} + \sum_{i=1}^{n} \log\left[1-(a)^2\right] - \sum_{i=1}^{n} \log\left[1-(1-\theta)(a)^2\right] - (\theta+1)\sum_{i=1}^{n} \frac{(a)^2}{1-(1-\theta)(a)^2} \rightarrow B$$

Again differentiate B with respect to the parameter θ

$$\frac{\partial^2 \log L}{\partial \theta^2} = \frac{-2n}{\theta^2} - \sum_{i=1}^{n} \frac{(a)^2\left[3e^{-x^2/2\sigma^2} + \theta(a)^2 - 1\right]}{\left[1-(1-\theta)(a)^2\right]^2}$$

Now partial differentiate A with parameter σ

$$\frac{\partial \log L}{\partial \sigma} = \sum_{i=1}^{n} \frac{x^2}{\sigma^3} - \sum_{i=1}^{n} \frac{e^{-x^2/2\sigma^2} x^2}{\sigma^3}\left[\frac{1}{a} - 2(a)\left[\frac{(\theta-1)}{\left[1-(a)^2\right]} - \frac{(1-\theta^2)}{\left[1-(1-\theta)(a)^2\right]}\right]\right] - \frac{2n}{\sigma} \rightarrow C$$

Again differentiate C with respect to the parameter σ

Now differentiate C with parameter θ

$$\frac{\partial^2 \log L}{\partial \theta \partial \sigma} = 0 - \sum_{i=1}^{n} \frac{e^{-x^2/2\sigma^2} x^2}{\sigma^3}\left[0 - \frac{2(a)}{\left[1-(a)^2\right]} + 2(a)\left[\frac{\left\{\left[1-(1-\theta)(a)^2\right][-2\theta] - (1-\theta^2)(a)^2\right\}}{\left[1-(1-\theta)(a)^2\right]^2}\right]\right] - 0$$

$$= \sum_{i=1}^{n} \frac{2e^{-x^2/2\sigma^2} x^2}{\sigma^3}\left[\frac{(a)}{\left(1-(a)^2\right)} - \frac{(a)\left(-2\theta - \left[(1-\theta)^2 (a)^2\right]\right)}{\left[1-(1-\theta)(a)^2\right]^2}\right]$$

Now differentiate B with parameter σ

$$\frac{\partial^2 \log L}{\partial \sigma \partial \theta} = \sum_{i=1}^{n} \frac{-2(a)\left(-e^{-x^2/2\sigma^2}\right)\left(\frac{x^2}{\sigma^3}\right)}{\left[1-(a)^2\right]} - \sum_{i=1}^{n} \frac{-2(1-\theta)(a)\left(-e^{-x^2/2\sigma^2}\right)\left(\frac{x^2}{\sigma^3}\right)}{\left[1-(1-\theta)(a)^2\right]} - (\theta+1)\sum_{i=1}^{n}\left\{\frac{\left[\left(1-(1-\theta)(a)^2\right)\right]\left[2(a)\left(-e^{-x^2/2\sigma^2}\right)\left(\frac{x^2}{\sigma^3}\right)\right] - \left[(a)^2\right]\left[-2(1-\theta)(a)\left(-e^{-x^2/2\sigma^2}\right)\left(\frac{x^2}{\sigma^3}\right)\right]}{\left[1-(1-\theta)(a)^2\right]^2}\right\}$$

$$= \sum_{i=1}^{n} \frac{2e^{-x^2/2\sigma^2} x^2}{\sigma^3}\left[\frac{(a)}{\left(1-(a)^2\right)} - \frac{(a)\left(-2\theta - \left[(1-\theta)^2 (a)^2\right]\right)}{\left[1-(1-\theta)(a)^2\right]^2}\right]$$

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